

Research Article

Mei Symmetry and Conservation Laws for Time-Scale Nonshifted Hamilton Equations

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The Mei symmetry and conservation laws for time-scale nonshifted Hamilton equations are explored, and the Mei symmetry theorem is presented and proved. Firstly, the time-scale Hamilton principle is established and extended to the nonconservative case. Based on the Hamilton principles, the dynamic equations of time-scale nonshifted constrained mechanical systems are derived. Secondly, for the time-scale nonshifted Hamilton equations, the definitions of Mei symmetry and their criterion equations are given. Thirdly, Mei symmetry theorems are proved, and the Mei-type conservation laws in time-scale phase space are driven. Two examples show the validity of the results.

1. Introduction

As we all know, the symmetry property of dynamic systems is the invariance of some physical quantity to the infinitesimal transformations of a group in mathematical form and can be expressed as a conservation law in physics [1–5]. The classical symmetries we are familiar with include Noether symmetry and Lie symmetry. Mathematically, the Noether symmetry is the invariance of the action functional under the infinitesimal transformations of a group, and physically, it is the Noether conservation laws [4, 5]. The Lie symmetry is the invariance of the differential equation under the infinitesimal transformations of a group [2, 6, 7]. Correspondingly, the Hojman conservation laws can be derived [5, 8, 9]. The Mei symmetry we are going to study was first proposed by Professor Mei in 2000 [10] and later popularized by many scholars [11–17]. Compared with Noether or Lie symmetry, Mei symmetry is a new kind of symmetry, and it refers to the invariance that the dynamic functions after infinitesimal transformation still make the dynamic equations hold. From Mei symmetry, a new kind of conservation law can be brought about, which is different from the Noether or Hojman one and called the Mei conserved quantity. The Mei symmetry theorem has been extended to fractional-order mechanical systems [18, 19] and nonstan-

dard Lagrangian dynamics [20]. Recently, we studied Mei symmetries on time scales [21, 22], but the research is preliminary and limited to Lagrange equations and Birkhoff equations.

The time scale, namely, any nonempty closed subset of the real number set, was first introduced by Dr. Hilger [23]. Since the real numbers and the integers themselves are special time scales, it is possible to deal with not only the continuous system and the discrete system uniformly but also the complex dynamic processes that have both continuity and discretization in time scales. In the past three decades, the time-scale analysis has been continuously improved and its application has been expanded [24–30]. Bartosiewicz and Torres presented Noether's theorem [31] and the second Euler-Lagrange equations [32] for the time-scale shifted Lagrange systems. Anerot and his collaborators [33] proved Noether's theorem in the time-scale version of Lagrange systems under the framework of shifted and nonshifted delta calculus of variation, and the results are also a correction of References [31, 32]. In the past decade, the study of time-scale dynamics and its symmetry has attracted extensive attention and made important progress, as shown in References [34–44]. However, the research is mainly limited to the following: (1) conservative system, (2) Noether symmetry, and (3) Noether-type conservation laws.

Furthermore, according to Bourdin's study [45], the results of the nonshifted case at the discrete level maintain the variational structure and related properties, although so far there have been few studies on the time-scale nonshifted variational problem.

This paper will focus on exploring Mei symmetry for time-scale nonshifted general holonomic systems and non-holonomic systems under the Hamiltonian framework, prove Mei symmetry theorems, and derive Mei conservation laws. The time-scale Mei symmetry is different from Noether or Lie symmetry on time scales. Mathematically, it is a new symmetry under the infinitesimal transformations of a group. Physically, it leads to a new kind of conservation law. The time-scale Mei symmetry not only unifies the Mei symmetry of the continuous case and the discrete case but also can obtain various discrete models due to the arbitrariness of time scales, which provides a new perspective for the structure-preserving algorithm of mechanical systems. Up to now, although there have been many research studies on Noether or Lie symmetry on time scales, the research on time-scale nonshifted Mei symmetry has not been reported.

2. Time-Scale Nonshifted Dynamic Equations

For time-scale calculus and its basic properties, the reader is recommended to refer to [24, 25].

2.1. Hamilton Principle and Its Extension. The time-scale nonshifted Hamilton action is

$$A = \int_{t_1}^{t_2} [p_i q_i^\Delta - H(t, q_i(t), p_i(t))] \Delta t, \quad (1)$$

where $H : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the Hamiltonian, $q_i^\Delta(t)$ is the generalized velocity, i.e., delta derivative of generalized coordinate $q_i(t)$ with respect to time t , and $p_i(t)$ is the generalized momentum, $i = 1, 2, \dots, n$. All functions are of $C_{rd}^{1,\Delta}(\mathbb{T})$. We refer to equation (1) as nonshifted Hamilton action in which the variables are q_i and p_i (instead of q_i^σ and p_i^σ or q_i^ρ and p_i^ρ) [45], where σ is the forward jump operator and ρ is the backward jump operator.

The variational principle

$$\delta A = 0, \quad (2)$$

with the endpoint conditions

$$\delta q_i|_{t=t_1} = \delta q_i|_{t=t_2} = 0, \quad (3)$$

and the commutative relation

$$\begin{aligned} \delta q_i^\Delta &= (\delta q_i)^\Delta, \\ \delta p_i^\Delta &= (\delta p_i)^\Delta, \end{aligned} \quad (4)$$

is called the time-scale nonshifted Hamilton principle.

Principle (2) can be generalized as

$$\int_{t_1}^{t_2} \{ \delta (p_i q_i^\Delta - H) + Q_i \delta q_i \} \Delta t = 0, \quad (5)$$

where $Q_i = Q_i(t, q_j, p_j)$ are nonpotential generalized forces. Principle (5) is the nonshifted Hamilton principle for time-scale general holonomic systems.

2.2. Hamiltonian System. From principle (2), it is easy to derive

$$\begin{aligned} q_i^\Delta &= \frac{\partial H}{\partial p_i}, \\ p_i^\nabla &= -\sigma^\nabla \frac{\partial H}{\partial q_i}. \end{aligned} \quad (6)$$

Equation (6) contains nonshifted Hamilton canonical equations for the time-scale Hamiltonian system.

If we take $\mathbb{T} = \mathbb{R}$, then the equations in (6) are reduced to the classical Hamilton canonical equations.

If we take $\mathbb{T} = \mathbb{Z}$, then the equations in (6) become

$$\begin{aligned} \Delta q_i^k &= \frac{\partial H(t_k, q_j^k, p_j^k)}{\partial p_i^k}, \\ \nabla p_i^k &= -\frac{\partial H(t_k, q_j^k, p_j^k)}{\partial q_i^k}, \end{aligned} \quad (7)$$

where Δ is the forward difference and ∇ is the backward difference.

2.3. General Holonomic System. From principle (5), it is easy to derive

$$\begin{aligned} q_i^\Delta &= \frac{\partial H}{\partial p_i}, \\ p_i^\nabla &= -\sigma^\nabla \frac{\partial H}{\partial q_i} + \sigma^\nabla Q_i. \end{aligned} \quad (8)$$

Equation (8) contains nonshifted Hamilton equations for the time-scale general holonomic system.

2.4. Nonholonomic System. The nonholonomic constraints are

$$\phi_\alpha(t, q_i, q_i^\Delta) = 0 \quad (\alpha = 1, 2, \dots, m). \quad (9)$$

The restriction applied by constraints in (9) on virtual displacements δq_i is

$$\Phi_{\alpha i}(t, q_j, q_j^\Delta) \delta q_i = 0. \quad (10)$$

If $\Phi_{\alpha i}$ is independent of $\partial\phi_{\alpha}/\partial q_i^{\Delta}$, then formula (9) represents non-Chetaev constraints, and if $\Phi_{\alpha i} = \partial\phi_{\alpha}/\partial q_i^{\Delta}$, then formula (9) represents Chetaev constraints.

Assume that the Lagrangian is $L = L(t, q_i, q_i^{\Delta})$, and let $p_i = \partial L/\partial q_i^{\Delta}$, and then $q_i^{\Delta} = q_i^{\Delta}(t, q_j, p_j)$. Thus, equations (9) and (10) can be written as

$$\phi_{\alpha}^*(t, q_i, p_i) = 0, \quad (11)$$

$$\Phi_{\alpha i}^*(t, q_j, p_j) \delta q_i = 0. \quad (12)$$

The nonshifted Hamilton equations can be expressed as

$$\begin{aligned} q_i^{\Delta} &= \frac{\partial H}{\partial p_i}, \\ p_i^{\nabla} &= -\sigma^{\nabla} \frac{\partial H}{\partial q_i} + \sigma^{\nabla} Q_i + \sigma^{\nabla} \lambda_{\alpha} \Phi_{\alpha i}^*. \end{aligned} \quad (13)$$

If the system is nondegenerate, the constraint multiplier λ_{α} can be solved by equations (11) and (13) as a function of generalized coordinates, generalized momentum, and time. Thus, equation (13) can be written as

$$\begin{aligned} q_i^{\Delta} &= \frac{\partial H}{\partial p_i}, \\ p_i^{\nabla} &= -\sigma^{\nabla} \frac{\partial H}{\partial q_i} + \sigma^{\nabla} (Q_i + \Pi_i), \end{aligned} \quad (14)$$

where $\Pi_i = \lambda_{\alpha} \Phi_{\alpha i}^*$ represent the constraining forces corresponding to the nonholonomic constraints. Equation (14) can be treated as a holonomic system corresponding to the nonholonomic system determined by (11) and (13). If constraint equation (11) is initially satisfied, then the motion of the nonholonomic system determined by (11) and (13) is solved from equation (14).

3. Mei Symmetry

3.1. Hamiltonian System. Introduce the infinitesimal transformations of the group:

$$\begin{aligned} \bar{t} &= t + v\xi_0(t, q_i, p_i), \\ \bar{q}_i(\bar{t}) &= q_i(t) + v\xi_i(t, q_j, p_j), \\ \bar{p}_i(\bar{t}) &= p_i(t) + v\zeta_i(t, q_j, p_j) \quad (i, j = 1, 2, \dots, n), \end{aligned} \quad (15)$$

where $v \in \mathbb{R}$ is a small parameter. Under the transformations in (15), the Hamiltonian H is transformed into \bar{H} , and then

$$\bar{H} = H(\bar{t}, \bar{q}_i(\bar{t}), \bar{p}_i(\bar{t})) = H(t, q_i(t), p_i(t)) + vY^{(0)}(H) + O(v^2), \quad (16)$$

where $Y^{(0)} = \xi_0 \partial/\partial t + \xi_i \partial/\partial q_i + \zeta_i \partial/\partial p_i$.

Definition 1. For the time-scale nonshifted Hamiltonian system (6), if

$$\begin{aligned} q_i^{\Delta} &= \frac{\partial \bar{H}}{\partial p_i}, \\ p_i^{\nabla} &= -\sigma^{\nabla} \frac{\partial \bar{H}}{\partial q_i}, \end{aligned} \quad (17)$$

hold, then the transformations in (15) are said to be Mei symmetric.

Criterion 2. If the transformations in (15) satisfy the following criterion equations

$$\begin{aligned} \frac{\partial Y^{(0)}(H)}{\partial p_i} &= 0, \\ -\sigma^{\nabla} \frac{\partial Y^{(0)}(H)}{\partial q_i} &= 0, \end{aligned} \quad (18)$$

then they correspond to the Mei symmetry of the time-scale Hamiltonian system (6).

3.2. General Holonomic System. Assume that the time-scale Hamiltonian H and the time-scale generalized forces Q_i undergo transformations in (15) to become \bar{H} and \bar{Q}_i , where

$$\begin{aligned} \bar{H} &= H(t, q_i, p_i) + vY^{(0)}(H) + O(v^2), \\ \bar{Q}_i &= Q_i(t, q_j, p_j) + vY^{(0)}(Q_i) + O(v^2), \end{aligned} \quad (19)$$

and then we have the following.

Definition 3. For the time-scale nonshifted general holonomic system (8), if

$$\begin{aligned} q_i^{\Delta} &= \frac{\partial \bar{H}}{\partial p_i}, \\ p_i^{\nabla} &= -\sigma^{\nabla} \frac{\partial \bar{H}}{\partial q_i} + \sigma^{\nabla} \bar{Q}_i, \end{aligned} \quad (20)$$

hold, then the transformations in (15) are said to be Mei symmetric.

Criterion 4. If the transformations in (15) satisfy the following criterion equations

$$\begin{aligned} \frac{\partial Y^{(0)}(H)}{\partial p_i} &= 0, \\ -\sigma^{\nabla} \frac{\partial Y^{(0)}(H)}{\partial q_i} + \sigma^{\nabla} Y^{(0)}(Q_i) &= 0, \end{aligned} \quad (21)$$

then they correspond to the Mei symmetry of the time-scale general holonomic system (8).

3.3. *Nonholonomic System.* Assume that the Hamiltonian H , the generalized forces Q_i , the constraining forces Π_i , and the constraints ϕ_α^* on time scales undergo transformations in (15) to become

$$\begin{aligned}\bar{H} &= H(t, q_i, p_i) + vY^{(0)}(H) + O(v^2), \\ \bar{Q}_i &= Q_i(t, q_j, p_j) + vY^{(0)}(Q_i) + O(v^2), \\ \bar{\Pi}_i &= \Pi_i(t, q_j, p_j) + vY^{(0)}(\Pi_i) + O(v^2), \\ \bar{\phi}_\alpha^* &= \phi_\alpha^*(t, q_j, p_j) + vY^{(0)}(\phi_\alpha^*) + O(v^2),\end{aligned}\quad (22)$$

and then we have the following.

Definition 5. For the time-scale corresponding holonomic system (14), if

$$\begin{aligned}q_i^\Delta &= \frac{\partial \bar{H}}{\partial p_i}, \\ p_i^\nabla &= -\sigma^\nabla \frac{\partial \bar{H}}{\partial q_i} + \sigma^\nabla (\bar{Q}_i + \bar{\Pi}_i),\end{aligned}\quad (23)$$

hold, then the transformations in (15) are said to be Mei symmetric.

Criterion 6. If the transformations in (15) satisfy the following criterion equations

$$\begin{aligned}\frac{\partial Y^{(0)}(H)}{\partial p_i} &= 0, \\ -\sigma^\nabla \frac{\partial Y^{(0)}(H)}{\partial q_i} + \sigma^\nabla [Y^{(0)}(Q_i) + Y^{(0)}(\Pi_i)] &= 0,\end{aligned}\quad (24)$$

then they correspond to the Mei symmetry of the time-scale corresponding holonomic system (14).

Definition 7. For the time-scale nonholonomic system determined by (11) and (13), if equation (23) and the following equation

$$\bar{\phi}_\alpha^* = \phi_\alpha^*(\bar{t}, \bar{q}_i(\bar{t}), \bar{p}_i(\bar{t})) = 0, \quad (25)$$

hold, then the transformations in (15) are said to be Mei symmetric.

Criterion 8. If the transformations in (15) satisfy criterion equation (24) and the following restriction equation

$$Y^{(0)}(\phi_\alpha^*) = 0, \quad (26)$$

then they correspond to the Mei symmetry of the time-scale nonholonomic system determined by (11) and (13).

4. Mei Symmetry Theorems

4.1. Hamiltonian System

Theorem 9. Assuming that the transformations in (15) satisfy criterion equation (18), then the time-scale nonshifted Hamiltonian system (6) has a new conservation law, such as

$$\begin{aligned}I_M &= \int_{t_1}^t \xi_0 \left\{ -\sigma^\nabla \frac{\partial Y^{(0)}(H)}{\partial t} + \frac{\nabla}{\nabla t} Y^{(0)}(H) \right\} \nabla t + Y^{(0)}(p_i) \xi_i^\sigma \\ &\quad - Y^{(0)}(H) \xi_0^\sigma + G_M^\sigma,\end{aligned}\quad (27)$$

where G_M is the gauge function that satisfies

$$\begin{aligned}\frac{\Delta}{\Delta t} [Y^{(0)}(p_i)]^\rho \xi_i + Y^{(0)}(p_i) \xi_i^\Delta - Y^{(0)}[Y^{(0)}(H)] \\ - Y^{(0)}(H) \xi_0^\Delta + G_M^\Delta = 0.\end{aligned}\quad (28)$$

Proof.

$$\begin{aligned}\frac{\nabla}{\nabla t} I_M &= -\xi_0 \sigma^\nabla \frac{\partial Y^{(0)}(H)}{\partial t} + \xi_0 \frac{\nabla}{\nabla t} Y^{(0)}(H) + \xi_i \frac{\nabla}{\nabla t} Y^{(0)}(p_i) \\ &\quad + \sigma^\nabla \xi_i^\Delta Y^{(0)}(p_i) - \xi_0 \frac{\nabla}{\nabla t} Y^{(0)}(H) - \sigma^\nabla \xi_0^\Delta Y^{(0)}(H) + \frac{\nabla}{\nabla t} G_M^\sigma \\ &= \sigma^\nabla \left\{ Y^{(0)}(p_i) \xi_i^\Delta + \frac{\Delta}{\Delta t} [Y^{(0)}(p_i)]^\rho \xi_i - Y^{(0)}[Y^{(0)}(H)] \right. \\ &\quad \left. - Y^{(0)}(H) \xi_0^\Delta + G_M^\Delta \right\} + \xi_i \sigma^\nabla \frac{\partial Y^{(0)}(H)}{\partial q_i} + \xi_i \sigma^\nabla \frac{\partial Y^{(0)}(H)}{\partial p_i}.\end{aligned}\quad (29)$$

Substituting equations (18) and (28) into (29), we get

$$\frac{\nabla}{\nabla t} I_M = 0. \quad (30)$$

Therefore, equation (27) is a conservation law. \square

Theorem 9 is the Mei symmetry theorem for the time-scale nonshifted Hamiltonian system (6), and equation (27) is called a Mei conservation law.

4.2. General Holonomic System

Theorem 10. Assuming that the transformations in (15) satisfy criterion equation (21), then the time-scale nonshifted general holonomic system (8) has a new conservation law, such as

$$\begin{aligned}I_M &= \int_{t_1}^t \xi_0 \left\{ -\sigma^\nabla Y^{(0)}(Q_i) q_i^\Delta - \sigma^\nabla \frac{\partial Y^{(0)}(H)}{\partial t} + \frac{\nabla}{\nabla t} Y^{(0)}(H) \right\} \nabla t \\ &\quad + Y^{(0)}(p_i) \xi_i^\sigma - Y^{(0)}(H) \xi_0^\sigma + G_M^\sigma,\end{aligned}\quad (31)$$

where G_M is the gauge function that satisfies

$$\begin{aligned} \frac{\Delta}{\Delta t} \left[Y^{(0)}(p_i) \right]^p \xi_i + Y^{(0)}(p_i) \xi_i^\Delta - Y^{(0)} \left[Y^{(0)}(H) \right] \\ - Y^{(0)}(H) \xi_0^\Delta + Y^{(0)}(Q_i) (\xi_i - q_i^\Delta \xi_0) + G_M^\Delta = 0. \end{aligned} \quad (32)$$

Proof.

$$\begin{aligned} \frac{\nabla}{\nabla t} I_M = & -\sigma^\nabla Y^{(0)}(Q_i) q_i^\Delta - \xi_0 \sigma^\nabla \frac{\partial Y^{(0)}(H)}{\partial t} + \xi_0 \frac{\nabla}{\nabla t} Y^{(0)}(H) \\ & + \xi_i \frac{\nabla}{\nabla t} Y^{(0)}(p_i) + \sigma^\nabla \xi_i^\Delta Y^{(0)}(p_i) - \xi_0 \frac{\nabla}{\nabla t} Y^{(0)}(H) \\ & - \sigma^\nabla \xi_0^\Delta Y^{(0)}(H) + \frac{\nabla}{\nabla t} G_M^\sigma \\ = & \sigma^\nabla \left\{ Y^{(0)}(p_i) \xi_i^\Delta + \frac{\Delta}{\Delta t} \left[Y^{(0)}(p_i) \right]^p \xi_i - Y^{(0)} \left[Y^{(0)}(H) \right] \right. \\ & \left. - Y^{(0)}(H) \xi_0^\Delta + G_M^\Delta + Y^{(0)}(Q_i) (\xi_i - q_i^\Delta \xi_0) \right\} \\ & + \xi_i \sigma^\nabla \left[\frac{\partial Y^{(0)}(H)}{\partial q_i} - Y^{(0)}(Q_i) \right] + \zeta_i \sigma^\nabla \frac{\partial Y^{(0)}(H)}{\partial p_i}. \end{aligned} \quad (33)$$

Substituting equations (21) and (32) into (33), we obtain $(\nabla/\nabla t)I_M = 0$. So the theorem is proved. \square

Theorem 10 is the Mei symmetry theorem for the time-scale nonshifted general holonomic system (8).

4.3. Nonholonomic System

Theorem 11. Assuming that the transformations in (15) satisfy criterion equation (24), then the time-scale corresponding holonomic system (14) has a new conservation law, such as

$$\begin{aligned} I_M = \int_{t_1}^t \xi_0 \left\{ -\sigma^\nabla \left[Y^{(0)}(Q_i) + Y^{(0)}(\Pi_i) \right] q_i^\Delta - \sigma^\nabla \frac{\partial Y^{(0)}(H)}{\partial t} + \frac{\nabla}{\nabla t} Y^{(0)}(H) \right\} \nabla t \\ + Y^{(0)}(p_i) \xi_i^\sigma - Y^{(0)}(H) \xi_0^\sigma + G_M^\sigma, \end{aligned} \quad (34)$$

where G_M is the gauge function that satisfies

$$\begin{aligned} \frac{\Delta}{\Delta t} \left[Y^{(0)}(p_i) \right]^p \xi_i + Y^{(0)}(p_i) \xi_i^\Delta - Y^{(0)} \left[Y^{(0)}(H) \right] - Y^{(0)}(H) \xi_0^\Delta \\ + \left[Y^{(0)}(Q_i) + Y^{(0)}(\Pi_i) \right] (\xi_i - q_i^\Delta \xi_0) + G_M^\Delta = 0. \end{aligned} \quad (35)$$

Theorem 12. Assuming that the transformations in (15) satisfy equations (24) and (26), then the time-scale nonholonomic system determined by (11) and (13) has a new conservation law (34), where the gauge function G_M satisfies equation (35).

We call Theorem 12 the Mei symmetry theorem for time-scale nonshifted nonholonomic systems determined by (11) and (13). Here, Theorem 11 establishes the relationship

between the Mei symmetry and the conservation law of time-scale Hamilton equations in (14).

5. Examples

Example 1. The time scale is $\mathbb{T} = \{2^m : m \in \mathbb{N}_0\}$, and the Hamiltonian is

$$H = \frac{1}{2} (p_1^2 + p_2^2) + q_1. \quad (36)$$

According to equation (6), we get

$$\begin{aligned} q_1^\Delta &= p_1, \\ q_2^\Delta &= p_2, \\ p_1^\nabla &= -\sigma^\nabla, \\ p_2^\nabla &= 0. \end{aligned} \quad (37)$$

To do the calculation, we have

$$Y^{(0)}(H) = \xi_1 + \zeta_1 p_1 + \zeta_2 p_2. \quad (38)$$

From equation (18), we obtain the following criterion equations:

$$\begin{aligned} \frac{\partial \xi_1}{\partial p_1} + \frac{\partial \zeta_1}{\partial p_1} p_1 + \zeta_1 + \frac{\partial \zeta_2}{\partial p_1} p_2 &= 0, \\ \frac{\partial \xi_1}{\partial p_2} + \frac{\partial \zeta_1}{\partial p_2} p_1 + \frac{\partial \zeta_2}{\partial p_2} p_2 + \zeta_2 &= 0, \\ \frac{\partial \xi_1}{\partial q_1} + \frac{\partial \zeta_1}{\partial q_1} p_1 + \frac{\partial \zeta_2}{\partial q_1} p_2 &= 0, \\ \frac{\partial \xi_1}{\partial q_2} + \frac{\partial \zeta_1}{\partial q_2} p_1 + \frac{\partial \zeta_2}{\partial q_2} p_2 &= 0. \end{aligned} \quad (39)$$

From equation (39), we have

$$\begin{aligned} \xi_0^1 &= 0, \\ \xi_1^1 &= -p_2, \\ \xi_2^1 &= p_1^p, \\ \zeta_1^1 &= 0, \\ \zeta_2^1 &= 1, \\ \xi_0^2 &= 0, \\ \xi_1^2 &= -p_1 - p_2, \\ \xi_2^2 &= p_1^p + p_2^p, \\ \zeta_1^2 &= 1, \\ \zeta_2^2 &= 1. \end{aligned} \quad (40)$$

Substituting (40) and (41) into equation (28) and using equation (37), we can get

$$\begin{aligned} G_M^1 &= t, \\ G_M^2 &= p_1 + p_2 + t. \end{aligned} \quad (42)$$

According to Theorem 9, we have

$$\begin{aligned} I_M^1 &= p_1 + \sigma(t) = \text{const.}, \\ I_M^2 &= p_1 + p_2 + \sigma(t) = \text{const.}, \end{aligned} \quad (43)$$

where $\sigma(t) = 2t$ for $\mathbb{T} = 2^{\mathbb{N}_0}$. These are the two Mei conserved quantities of the system.

Example 2. We investigate a time-scale nonholonomic system whose Lagrangian function is

$$L = \frac{1}{2} \left[(q_1^\Delta)^2 + (q_2^\Delta)^2 + (q_3^\Delta)^2 \right] - q_1, \quad (44)$$

and the constraint equation is

$$\phi = q_2^\Delta - tq_1^\Delta = 0, \quad (45)$$

and the virtual displacements of the system satisfy

$$\delta q_1 - \delta q_2 = 0. \quad (46)$$

The generalized momentum and the Hamiltonian are

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial q_1^\Delta} = q_1^\Delta, \\ p_2 &= \frac{\partial L}{\partial q_2^\Delta} = q_2^\Delta, \\ p_3 &= \frac{\partial L}{\partial q_3^\Delta} = q_3^\Delta, \\ H &= \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + q_1. \end{aligned} \quad (47)$$

In canonical coordinates, constraint equation (45) becomes

$$\phi^* = p_2 - tp_1 = 0. \quad (48)$$

The time-scale dynamic equations are

$$\begin{aligned} q_1^\Delta &= p_1, \\ q_2^\Delta &= p_2, \\ q_3^\Delta &= p_3, \\ p_1^\nabla &= -\sigma^\nabla + \sigma^\nabla \lambda, \\ p_2^\nabla &= -\sigma^\nabla \lambda, \\ p_3^\nabla &= 0. \end{aligned} \quad (49)$$

From equations (48) and (49), we can get

$$\lambda = \frac{t^\rho \sigma^\nabla - p_1}{\sigma^\nabla (1 + t^\rho)}, \quad (50)$$

and therefore, the nonholonomic constraining forces are

$$\begin{aligned} \Pi_1 &= \frac{t^\rho \sigma^\nabla - p_1}{\sigma^\nabla (1 + t^\rho)}, \\ \Pi_2 &= -\frac{t^\rho \sigma^\nabla - p_1}{\sigma^\nabla (1 + t^\rho)}, \\ \Pi_3 &= 0. \end{aligned} \quad (51)$$

Thus, equations in (49) become

$$\begin{aligned} q_1^\Delta &= p_1, \\ q_2^\Delta &= p_2, \\ q_3^\Delta &= p_3, \\ p_1^\nabla &= -\sigma^\nabla + \frac{t^\rho \sigma^\nabla - p_1}{1 + t^\rho}, \\ p_2^\nabla &= -\frac{t^\rho \sigma^\nabla - p_1}{1 + t^\rho}, \\ p_3^\nabla &= 0. \end{aligned} \quad (52)$$

To do the calculation, we obtain

$$\begin{aligned} Y^{(0)}(H) &= \xi_1 + \zeta_1 p_1 + \zeta_2 p_2 + \zeta_3 p_3, \\ Y^{(0)}(\Pi_1) &= -Y^{(0)}(\Pi_2) = \xi_0 \frac{\partial}{\partial t} \left[\frac{t^\rho \sigma^\nabla - p_1}{\sigma^\nabla (1 + t^\rho)} \right] - \frac{\zeta_1}{\sigma^\nabla (1 + t^\rho)}, \\ Y^{(0)}(\Pi_3) &= 0, \\ Y^{(0)}(\phi^*) &= \zeta_2 - \xi_0 p_1 - t\zeta_1. \end{aligned} \quad (53)$$

Take the generators as

$$\begin{aligned} \xi_0 &= 0, \\ \xi_1 &= -p_2 + p_3, \\ \xi_2 &= p_1^\rho + p_2^\rho, \\ \xi_3 &= p_3^\rho, \\ \zeta_1 &= 0, \\ \zeta_2 &= 1, \\ \zeta_3 &= -1. \end{aligned} \quad (54)$$

The generators in (54) satisfy equation (24), according to Criterion 6, which correspond to the Mei symmetry of the corresponding holonomic system (52). However, the generators in (54) do not satisfy $Y^{(0)}(\phi^*) = 0$, so they are not the Mei symmetry of the nonholonomic system determined by

(48) and (49). Substituting equation (54) into equation (35), we solve

$$G_M = t. \quad (55)$$

So from Theorem 11, we have

$$I_M = p_1 + p_2 - p_3 + \sigma(t) = \text{const}. \quad (56)$$

This is the Mei conserved quantity of the corresponding nonholonomic system (52).

If we take the generators as

$$\begin{aligned} \xi_0 &= 0, \\ \xi_1 &= -p_3, \\ \xi_2 &= 0, \\ \xi_3 &= p_3^p + t, \\ \zeta_1 &= 0, \\ \zeta_2 &= 0, \\ \zeta_3 &= 1. \end{aligned} \quad (57)$$

According to Criterion 8, the generators in (57) correspond to the Mei symmetry of the nonholonomic system determined by (48) and (49). Substituting equation (57) into equation (35), we get

$$G_M = -t. \quad (58)$$

From Theorem 12, we have

$$I_M = p_3 = \text{const}. \quad (59)$$

6. Conclusions

The study of symmetry and conservation laws has always been an important aspect of analytical mechanics. Noether symmetry and Lie symmetry are two different classical symmetries. Mei symmetry is essentially different from these two, and it can directly yield a Mei conservation law. In this paper, we proved Mei symmetry theorems for time-scale nonshifted mechanical systems under the Hamiltonian framework. The main work of this paper is to extend Mei's concept of form invariance to arbitrary time-scale frameworks and obtain the new conservation laws for time-scale nonshifted mechanical systems. The details are as follows.

Firstly, we proposed the time-scale Hamilton principles (2) and (5) and derived the time-scale dynamic equations (6), (8), and (13). Secondly, we defined Mei symmetry for time-scale mechanical systems under the Hamiltonian framework and derived its criterion equations. Thirdly, we proposed and proved Mei symmetry theorems for time-scale mechanical systems under the Hamiltonian framework and obtained the Mei conservation laws (27), (31), and (34).

If we take $\mathbb{T} = \mathbb{R}$, then we have $\sigma(t) = t$ and $\mu(t) = 0$, so the above results are reduced to the continuous versions of

the variational principle, dynamic equations, and Mei symmetry theorems under the Hamiltonian framework.

If we take $\mathbb{T} = h\mathbb{Z}$, then we have $\sigma(t) = t + h$ and $\mu(t) = h$, so the above results are reduced to the discrete versions of the variational principle, dynamic equations, and Mei symmetry theorems with step size h .

Since there are many alternatives to real and integer numbers on the time scale, our results are general.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there is no conflict of interest in the publication of this article.

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